SUBSPACES

EXAMPLE: Suppose $(X,\mathcal{T})$ is a topological space and A is a nonempty subset of X. Let $\mathcal{T}_A = \{A\cap T\ :\ T\in\mathcal{T}\}$.

Prove $\mathcal{T}_A$ is a topology on A.

NOTE: In this case $\mathcal{T}_A$ is the **subspace topology** on A induced by $\mathcal{T}$. If $U\in\mathcal{T}_A$, we say U is **open** in A.

EXAMPLE: Let $(\mathbb{R},\mathcal{E})$ be the real numbers with the Euclidean Topology.

- Show $[0,1)$ is open in $[0,2]$ but $[0,1)$ is not open in $\mathbb{R}$.
- Is $\{0\}$ open in $\mathbb{Q}$? Explain.
- Find a subspace of $\mathbb{R}$ which is disconnected.

EXAMPLE: Suppose A is a nonempty subset of X.

If $\mathcal{B}$ is a base for a topology $\mathcal{T}$ on X, show $\mathcal{B}_A = \{A\cap B\,|\,B\in\mathcal{B}\}$ is a base for $\mathcal{T}_A$.

EXAMPLE: Suppose $(X,\mathcal{T})$ is a topological space with subspace $(A,\mathcal{T}_A)$. Prove or disprove:

$$D \text{ is closed in } A \text{ iff } D = A\cap C \text{ where } C \text{ is closed in } X.$$

DEFINITION: An **interval** is a subset $I\subseteq\mathbb{R}$ such that if $a,b\in I$ with $a<b$, then $c\in I$ for all c with $a<c<b$.

EXAMPLE: Show that using the above definition, $\emptyset$ and $\{x\}$ are intervals.

THEOREM: Let $(\mathbb{R},\mathcal{E})$ be the real numbers with the Euclidean Topology.

$$A\subseteq\mathbb{R} \text{ is connected iff } A \text{ is an interval.}$$

EXAMPLE: Prove if A and B are connected subspaces of $(X,\mathcal{T})$ and $A\cap B\neq\emptyset$, then $A\cup B$ is connected.

DEFINITION: Suppose A is a nonempty subset of X and $F: X\to Y$ is a function.

We define the **restriction** of F to A, $F|_A : A\to Y$ as $F|_A(a) = F(a)$ for all $a\in A$. $F|_A$ is a function (why?)

EXAMPLE: Suppose A is a nonempty subset of X and $F:(X,\mathcal{T})\to(Y,\mathcal{U})$ is continuous.

Prove if $F:(X,\mathcal{T})\to(Y,\mathcal{U})$ is continuous, then so is $F|_A:(A,\mathcal{T}_A)\to(Y,\mathcal{U})$

Said differently, restrictions of continuous functions are continuous.

EXAMPLE: Suppose $F:(X,\mathcal{T})\to(Y,\mathcal{U})$. Let $B=F(X)$. Then:

$$F:(X,\mathcal{T})\to(Y,\mathcal{U}) \text{ is continuous iff } F:(X,\mathcal{T})\to(B,\mathcal{U}_B) \text{ is continuous.}$$

In other words, it is only the topology on the **range** which determines continuity.

EXAMPLE: Prove continuous images of connected spaces are connected. More precisely, show:

$$\text{If } (X,\mathcal{T}) \text{ is connected and } F:(X,\mathcal{T})\to(Y,\mathcal{U}) \text{ is continuous, then } F(X) \text{ is connected.}$$

EXAMPLE: Prove the Intermediate Value Theorem.